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Conformal Einstein spaces in *N*-dimensions: Part II

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Abstract

This paper generalizes the main result of "Conformal Einstein spaces in *N*-dimensions" published in *Ann. Global Anal. Geom.* 20(2) (2001). We present necessary and sufficient tensorial conditions for a certain class of semi-Riemannian manifolds to be conformally related to Einstein spaces.

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1. Introduction

Riemannian spaces conformally related to Einstein spaces were already considered by Brinkmann in the 1920s (cf. [2,3]). Because of the interest in theoretical physics this problem is often studied in four-dimensional Lorentz geometry (cf. [19,12,20,4,11]). The results of Kozameh et al. [12], Wünsch [20] and Czapor et al. [4] provide tensorial invariants which vanish if and only if the four-dimensional Lorentz manifold is conformally related to an Einstein space. In [15], we found necessary and sufficient conditions for *N*-dimensional semi-Riemannian manifolds to be conformally related to Einstein spaces under the assumption that $\det \mathcal{W} \neq 0$, where $\mathcal{W} : \Lambda^2(T^*M) \to \Lambda^2(T^*M)$ is the Weyl tensor considered as

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endomorphism on two forms. Recently Edgar [7] gave an explicit expression for the considered vector field in [15] which makes the computations easier. Furthermore, Gover and Nurowski [9] improved the result for the N-dimensional case. They provided necessary and sufficient tensorial conditions for weakly generic metrics to be conformal Einstein spaces where weakly generic means that the rank of the below introduced bundle $\mathcal{E}$ vanishes at all points of the manifold: $\mathrm{rk}(\mathcal{E}) = 0$. The main results in this paper deal with the case that the rank of $\mathcal{E}$ is less or equal to one on an open and dense subset of the manifold. Since $\det \mathcal{W} \neq 0$ yields $\mathrm{rk}(\mathcal{E}) = 0$, but on the contrary there is a huge variety of manifolds with $\mathrm{rk}(\mathcal{E}) \leq 1$ and $\det \mathcal{W} = 0$, the results in this paper and in [9] mean a major improvement to the result in [15].

Suppose (M, g) is a semi-Riemannian manifold of dimension $n \geq 4$ and W the Weyl tensor, then the set

$$\mathcal{E} := \{v \in TM | W(v, ., ., .) = 0\} \subseteq TM$$

is conformally invariant and a vector space in each fiber. Let $M_{\mathcal{E}}$ be the set of all points of M at which the rank of $\mathcal{E}$ is locally constant, then $M_{\mathcal{E}}$ is open and dense in M and $\mathcal{E}$ is a vector bundle over $M_{\mathcal{E}}$. In Section 4, we prove some basic properties of $\mathcal{E}$, for instance that $\mathcal{E}$ is integrable on spaces with harmonic Weyl tensor (C-spaces). Another fact is that the rank of $\mathcal{E}$ is less or equal to $n - 4$ as long as g is not conformally flat and the restriction of g to $\mathcal{E}$ is non-degenerate. In particular, in the four-dimensional Riemannian case each connected component U of $M_{\mathcal{E}}$ is either conformally flat (i.e. $\mathcal{E} = TM$) or $\mathcal{E}$ is trivial on U (i.e. $\mathcal{E} = \{0\}$). As already mentioned, metrics with $\mathcal{E} = \{0\}$ are called weakly generic in [9].

Since Einstein spaces are C-spaces and the conformal equivalence problem for C-spaces is a linear problem, we start with the consideration of conformal C-spaces. A semi-Riemannian manifold (M, g) is conformally related to a C-space if there exists a function $\phi : M \to \mathbb{R}$ with

$$0 = \delta W - (n - 3)W(., ., ., \nabla\phi).$$

The solution $\nabla\phi$ of this equation is unique in $TM/\mathcal{E}$. In Section 5, we define explicitly a smooth vector field $\mathbb{T} : M_{\mathcal{E}} \to TM$ which is the only possible solution of the above equation in $TM/\mathcal{E}$. In particular, a semi-Riemannian manifold is conformally related to a C-space if and only if

$$C_{\mathbb{T}} = \frac{1}{n-3}\delta W - W(., ., ., \mathbb{T})$$

vanishes and there is a vector field $V : M_{\mathcal{E}} \to \mathcal{E}$ such that $V + \mathbb{T}$ is a gradient field on all of M. The vector field $\mathbb{T}$ is well defined for all semi-Riemannian manifolds (M, g), but generally depends on a choice of a Riemannian metric h on M. In order to define $\mathbb{T}$, we show in Section 3 that the Moore–Penrose inverse (taken with respect to a Riemannian metric) of a smooth endomorphism yields a smooth endomorphism on an open and dense subset of the manifold, in our case this subset coincides with $M_{\mathcal{E}}$.

In Section 6, we consider spaces conformally related to Einstein spaces of dimension $n \geq 4$, since the three-dimensional case is trivial. We prove that the $(0, 2)$ tensor

$$E_{\mathbb{T}} = \mathrm{Ric} - \frac{\mathrm{scal}}{n} g + (n-2)\left(\nabla \mathbb{T}^* + \mathbb{T}^* \otimes \mathbb{T}^* - \frac{1}{n}(\mathrm{div}(\mathbb{T}) + |\mathbb{T}|^2) g\right)$$

is conformally invariant in case $\mathrm{rk}(\mathcal{E}) = 0$. Hence, a semi-Riemannian manifold (M, g) with $\mathcal{E} = \{0\}$ is locally conformally related to an Einstein space if and only if $E_{\mathbb{T}}$ vanishes. A similar statement was also shown by Gover and Nurowski in [9] (weakly generic case). This is a major extension of the previous result in [15], since $\det \mathcal{W} \neq 0$ yields $\mathrm{rk}(\mathcal{E}) = 0$, but there is a huge variety of spaces with $\mathrm{rk}(\mathcal{E}) = 0$ and $\det \mathcal{W} = 0$. For instance, all four-dimensional Riemannian manifolds which are (anti) self-dual and not conformally flat satisfy $\mathrm{rk}(\mathcal{E}) = 0$ and $\det \mathcal{W} = 0$.

In the last section, we discuss the case $\mathrm{rk}(\mathcal{E}) = 1$. In order to do so we suppose that V is a (normalized) nowhere vanishing vector field with values in $\mathcal{E}$ and consider the vector field $S = \mathbb{T} + fV$ for a function f. Trying to solve $E_S = 0$ supplies a first order pde-system of Riccati type for f. In most cases, it is possible to compute f explicitly by tensorial obstructions. However, if the Einstein metric is not unique in the conformal class, this system has two solutions. Moreover, this approach provides necessary and sufficient tensorial condition for the existence of a non-trivial solution ψ of $\nabla^2\psi = \frac{\triangle\psi}{n} g$ on Einstein spaces. Solutions of this pde-system characterize the different Einstein metrics in the conformal class as well as warped product metrics (cf. [13]), in the four-dimensional Lorentz case they characterize plane gravitational waves.

2. Preliminaries

Let (M^n, g) be a semi-Riemannian manifold and ∇ be the Levi–Civita connection for g, then R denotes the Riemannian curvature tensor of g:

$$R(X, Y, Z, T) = g(R_{X,Y}Z, T) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, T).$$

The Ricci tensor Ric is given by $\mathrm{Ric}(X, Y) = \mathrm{trace}\{V \mapsto R_{V,X}Y\}$ and the scalar curvature by $\mathrm{scal} = \mathrm{trace}(\mathrm{Ric})$. Using the Kulkarni–Nomizu product:

$$\begin{aligned}&(g \odot h)(X, Y, Z, T)\\ &\quad := g(X, T)h(Y, Z) + g(Y, Z)h(X, T) - g(X, Z)h(Y, T) - g(Y, T)h(X, Z)\end{aligned}$$

we obtain the Weyl tensor W and the Schouten tensor $\mathfrak{k}$:

$$W := R - g \odot \mathfrak{k}, \qquad \mathfrak{k} := \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{scal}}{2(n-1)} g\right). \tag{1}$$

We consider conformal transformations $(M, g) \to (M, \bar{g} := \psi^{-2} g)$ and denote the symbols for $\bar{g}$ by $\bar{\nabla}, \bar{R}, \bar{W}, \ldots$. If $(M, g) \to (M, \bar{g} := \psi^{-2} g)$ is a conformal transformation with $\psi = \mathrm{e}^{\phi}$, the Levi–Civita connections and the Weyl tensors are related by:

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y - \mathrm{d}\phi(X)Y - \mathrm{d}\phi(Y)X + \langle X, Y\rangle \nabla^g \phi\\ \bar{W} &= \psi^{-2} W.\end{aligned} \tag{2}$$

In this case $\nabla^g \phi$ is the gradient of ϕ with respect to the metric g, in particular if $* : TM \to T^*M$ is the isomorphism given by $Y^*(X) = g(X, Y)$, $\mathrm{d}\phi$ equals $(\nabla^g \phi)^*$. We define the divergence of a $(0, s)$ tensor A by:

$$\delta_r(A) = \mathfrak{C}_{r,s+1}(\nabla A), \quad (r \le s)$$

where $\mathfrak{C}_{r,s}$ is the metric contraction with respect to the indices (r, s) and ∇A is the following $(0, s+1)$ tensor field:

$$(\nabla A)(X_1, \ldots, X_s, V) := (\nabla_V A)(X_1, \ldots, X_s).$$

If A is symmetric, $\delta_r A$ does not depend on r, and if A is a curvature operator, we introduce as abbreviation $\delta A := \delta_4 A$. Moreover, we define the exterior derivative of a symmetric $(0, 2)$ tensor b by

$$\mathrm{d}^\nabla b(X, Y, Z) := (\nabla_X b)(Y, Z) - (\nabla_Z b)(X, Z).$$

The differential Bianchi identity yields (cf. [1]; Ch. 16.3)

$$\delta R = \mathrm{d}^\nabla \mathrm{Ric} \qquad \text{and} \qquad \delta W = (n-3)\mathrm{d}^\nabla \mathfrak{k} \tag{3}$$

where $\mathfrak{k}$ is the Schouten tensor introduced in equation (1). A semi-Riemannian manifold (M^n, g) of dimension $n \ge 4$ is called *C-space* or *space with harmonic Weyl tensor* (cf. [1]; 16.D) if δW vanishes. Suppose $\overline{g} = \psi^{-2} g$ is a conformal transformation with $\psi = \mathrm{e}^\phi$, then we obtain the well-known relation (cf. [1]; 16.25):

$$\overline{\delta W} = \delta W - (n-3) W(., ., ., \nabla^g \phi). \tag{4}$$

3. Moore–Penrose inverse transformations

Suppose $(\mathcal{V}, h)$ is a Riemannian vector bundle over M. If $A : \mathcal{V} \to \mathcal{V}$ is a symmetric bundle endomorphism with respect to the inner product h, then there is at each point $p \in M$ an unique endomorphism $A_p^\# \in \mathrm{End}(\mathcal{V}_p)$ which satisfies (Moore–Penrose inverse; cf. [8])

$$\begin{aligned} &A_p \circ A_p^\# \circ A_p = A_p, \qquad A_p^\# \circ A_p \circ A_p^\# = A_p^\#, \\ &A_p \circ A_p^\# \quad \text{and} \quad A_p^\# \circ A_p \text{ are symmetric w.r.t. } h. \end{aligned} \tag{5}$$

Let M_A consist of all points $p \in M$ at which the number of distinct eigenvalues of A is locally constant, then M_A is open and dense in M. If $A \in \Gamma(\mathrm{End}(\mathcal{V}))$ is of order C^k, the map

$$A^\# : M_A \to \mathrm{End}(\mathcal{V}_{|M_A}), \quad p \mapsto A_p^\#$$

is of order C^k, in particular $A^\#$ is a section in $\mathrm{End}(\mathcal{V}_{|M_A})$. $A^\#$ is called the *Moore–Penrose inverse* of A. The bundle $\mathcal{V}$ admits in every point of M an orthogonal decomposition into eigenspaces of A: $\mathcal{V}_p = \mathcal{V}_p^1 \oplus \cdots \oplus \mathcal{V}_p^k$. Since the number of distinct eigenvalues of A is constant in the connected components of M_A, $p \mapsto \mathcal{V}^j_p$ is differentiable on each component of M_A. Thus, $\mathcal{V}^j \to U$ is a subbundle of $\mathcal{V} \to U$ as long as $U \subset M$ is contained in a connected component of M_A. The existence and the uniqueness of the Moore–Penrose inverse follows from linear algebra, while the differentiability of $A^\#$ can be proved as follows. Let U be a connected component of M_A, then $\mathrm{Im}(A) \to U$ and $\ker(A) \to U$ are subbundles of $\mathcal{V} \to U$. In particular, $A_{|U}$ supplies an orthogonal decomposition $\mathcal{V} = \mathrm{Im}(A) \oplus \ker(A)$. Now A restricted to $\mathrm{Im}(A)$ is invertible. Hence, if we set $A^\# := A^{-1}$ on $\mathrm{Im}(A)$ and $A^\# := 0$ on $\ker(A)$, the endomorphism $A^\#$ satisfies the identities in (5). Since $\mathrm{Im}(A)$ and $\ker(A)$ are subbundles of $\mathcal{V}_{|U}$ as well as the assignment $A \mapsto A^{-1}$ is smooth, $A^\#$ is of order C^k on M_A as long as A is of order C^k. Furthermore, if $\det A$ does not vanish on M, $A^\#$ equals A^{-1} and is defined on all of M.

Definition 1. Let $\mathfrak{T}^r_s(M) = (TM)^r \otimes (T^*M)^s$ denote the bundle of (r, s) tensor fields on a semi-Riemannian manifold (M, g). Suppose $A \in \Gamma(\mathfrak{T}^0_4(M))$ is an algebraic curvature tensor, then A becomes an endomorphism of $\mathfrak{T}^0_2(M)$ in the following way:

$$\mathcal{A} : \mathfrak{T}^0_2(M) \to \mathfrak{T}^0_2(M), \quad X^* \otimes Y^* \mapsto \mathcal{A}(X^* \otimes Y^*)$$

where $\mathcal{A}(X^* \otimes Y^*)(Z, T) := A(Y, Z, T, X)$. If $b \in \mathfrak{T}^0_2(M)$ is (skew) symmetric, $\mathcal{A}(b)$ is (skew) symmetric. In particular, $\mathcal{A}$ is an endomorphism on two forms:

$$\mathcal{A} : \Lambda^2(T^*M) \to \Lambda^2(T^*M).$$

Since the first Bianchi identity implies

$$g(\mathcal{A}(X^* \wedge Y^*), Z^* \wedge T^*) = \mathcal{A}(X^* \wedge Y^*)(Z, T) = A(X, Y, Z, T),$$

the endomorphism $\mathcal{A}$ is symmetric on $\Lambda^2(T^*M)$ with respect to the tensor product extension of g. In this case $X^* \wedge Y^*$ is the two form given by $X^* \otimes Y^* - Y^* \otimes X^*$.

Let W be the Weyl tensor on (M^n, g) and define

$$\mathcal{E} := \bigcup_{p \in M} \mathcal{E}_p \subseteq TM, \quad \text{where} \quad \mathcal{E}_p := \{v \in T_pM \,|\, W(v, ., ., .) = 0\}.$$

The set $\mathcal{E}_p$ is a vector space for all $p \in M$, but in general $\mathcal{E}$ is not a vector bundle over M. In the Riemannian case, the Weyl tensor considered as symmetric endomorphism $\mathcal{W} : \Lambda^2(T^*M) \to \Lambda^2(T^*M)$ is diagonalizable, however, if g is indefinite, the Weyl endomorphism $\mathcal{W}$ can be nilpotent. For instance, $\mathcal{W}$ is not diagonalizable for space times of Petrov type N. Thus, we introduce a Riemannian metric h on TM and extend h in the usual way to T^*M and $\Lambda^2(T^*M)$. Denote by $\mathcal{W}^{\mathrm{t}} : \Lambda^2(T^*M) \to \Lambda^2(T^*M)$ the adjoint of $\mathcal{W}$ with respect to h, then $\mathcal{W}^{\mathrm{t}}\mathcal{W}$ is non-negative definite and symmetric with respect to h. Moreover, let $\mathcal{E}^{\perp h}$ be the h-orthogonal complement of $\mathcal{E}$ in TM, then the h-orthogonal decomposition

$\Lambda^2(T^*M) = \ker(\mathcal{W}^{\mathrm{t}}\mathcal{W}) \oplus \mathrm{Im}(\mathcal{W}^{\mathrm{t}}\mathcal{W})$,

$$\Lambda^2(T_pM) = \Lambda^2(\mathcal{E}_p^{\perp h}) \oplus \Lambda^2(\mathcal{E}_p) \oplus \mathcal{E}_p^{\perp h} \otimes \mathcal{E}_p$$

and $\ker(\mathcal{W}^{\mathrm{t}}\mathcal{W}) = \ker(\mathcal{W})$ imply:

$$\dim\ker(\mathcal{W}_p) \geq \dim\mathcal{E}_p\left(n - \frac{\dim\mathcal{E}_p + 1}{2}\right). \tag{6}$$

In particular, this inequality shows that $\mathcal{E}_p$ is the trivial vector space if $\det\mathcal{W}_p$ does not vanish. Contrary, a lower bound of $\dim\mathcal{E}_p$ in terms of $\dim\ker(\mathcal{W}_p)$ is impossible in general. Since if M is even-dimensional and there is a non-degenerate two form η in the image of $\mathcal{W}_p$, the one-form $\eta(v, .)$ does not vanish for all $v \in T_pM - \{0\}$ which shows $\dim\mathcal{E}_p = 0$. In particular, $\mathrm{Im}(\mathcal{W}_p)$ could be two-dimensional ($\mathcal{W}$ is trace free) and $\dim\mathcal{E}_p = 0$.

Denote by $\mathfrak{w} : T^*M \to T^*M$ the negative Ricci contraction of $\mathcal{W}^{\mathrm{t}}\mathcal{W}$, i.e. if θ is a one-form and X a vector field, $\mathfrak{w}(\theta)(X)$ is the trace of the endomorphism:

$$T^*M \to T^*M, \qquad \eta \mapsto X \llcorner \mathcal{W}^{\mathrm{t}}\mathcal{W}(\theta \wedge \eta).$$

If $f_1, \ldots, f_n$ is a base of T_pM and $\eta_1, \ldots, \eta_n$ the corresponding cobase of T_p^*M ($\eta_j(f_i) = \delta_{ij}$), we have:

$$\mathfrak{w}(\theta) = \sum_{i=1}^{n} f_i \llcorner \mathcal{W}^{\mathrm{t}}\mathcal{W}(\eta_i \wedge \theta).$$

Using a h-orthonormal base, we conclude that $\mathfrak{w}$ is symmetric with respect to h and in particular, $\mathcal{W}^{\mathrm{t}}\mathcal{W} \geq 0$ supplies $\mathfrak{w} \geq 0$. Since $\mathfrak{w}$ is symmetric with respect to the positive definite metric h on T^*M, there is an open and dense subset $M_{\mathfrak{w}}$ of M on which the Moore–Penrose inverse of $\mathfrak{w}$ exists. Moreover, the isomorphism $* : TM \to T^*M$, $v \mapsto g(v, .)$ yields an isomorphism

$$\mathcal{E}_p \to \ker(\mathfrak{w}_p). \tag{7}$$

The fact $*(\mathcal{E}_p) \subseteq \ker(\mathfrak{w}_p)$ follows from the definitions. In order to see equality, let $v \in T_pM$ be a vector with $\mathfrak{w}(v^*) = 0$. We show $\mathcal{W}(v^* \wedge \theta) = 0$ which is equivalent to $\mathcal{W}^{\mathrm{t}}\mathcal{W}(v^* \wedge \theta) = 0$ for all $\theta \in T_p^*M$. Suppose $\eta_1, \ldots, \eta_m \in \Lambda^2(T_p^*M)$ is a h-orthonormal base of $\Lambda^2(T_p^*M)$ consisting of eigenvectors to the non-negative eigenvalues $\lambda_1, \ldots, \lambda_m$ of $\mathcal{W}^{\mathrm{t}}\mathcal{W}_p$. Considering the two forms η_j as skew symmetric maps $T_p^*M \to T_p^*M$ (use h) leads to $\mathfrak{w}_p = -\sum_{j=1}^{m} \lambda_j(\eta_j)^2$. Since each of these summands is non-negative definite, $\mathfrak{w}(v^*) = 0$ implies $(\eta_j)^2(v^*) = 0$ for all j with $\lambda_j \neq 0$. But this gives the claim $\mathcal{W}^{\mathrm{t}}\mathcal{W}(v^* \wedge \theta) = 0$. Therefore, equation (7) and the above arguments prove that $\mathcal{E} \to M_{\mathfrak{w}}$ is smooth. In particular, if U is a connected component of $M_{\mathfrak{w}}$, $\mathcal{E}_{|U}$ is a subbundle of $TM_{|U}$. In the introduction we defined $M_{\mathcal{E}}$ to be the set of all points $p \in M$ at which the rank of $\mathcal{E}$ is locally constant, in particular $M_{\mathfrak{w}} \subseteq M_{\mathcal{E}}$. Since the map $\mathbb{1} - \mathfrak{w}^{\#}\mathfrak{w}$ is the h-orthogonal projection $T^*M \to \ker(\mathfrak{w})$, the endomorphism $*^{-1}(\mathbb{1} - \mathfrak{w}^{\#}\mathfrak{w})*$ is the $h^* = g^{-1}hg$-orthogonal projection $TM \to \mathcal{E}$ on $M_{\mathfrak{w}}$ (in this case and in the following the Riemannian metric h^*

is defined on TM by $h^*(X, Y) := h(X^*, Y^*)$). Hence, $*^{-1}(\mathbb{1} - \mathfrak{w}^\#\mathfrak{w})* : TM_{\mathfrak{w}} \to \mathcal{E}$ can be differentiable extended to $M_{\mathcal{E}}$. Use (7) and the fact $\mathfrak{w}^\# = (\mathfrak{w}_{|\mathrm{Im}(\mathfrak{w})})^{-1}$ to show that $\mathfrak{w}^\#$ is differentiable on $M_{\mathcal{E}}$. Thus, we can assume $M_{\mathfrak{w}} = M_{\mathcal{E}}$, while $M_{\mathcal{E}}$ does not depend on the choice of the Riemannian metric h.

Since the Weyl tensor is conformally invariant, it should be mentioned that $\mathcal{E}$, $\ker(\mathcal{W})$ and $M_{\mathcal{E}}$ are invariant under conformal transformation.

Example 2. The Reissner–Nordström solution (cf. [10]; (5.5)) provides an example of a manifold with $M_{\mathcal{E}} \neq M$. Let:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{(1 - \frac{2m}{r} + \frac{e^2}{r^2})} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

be the line element on $M := \mathbb{R} \times \mathbb{R}^{>0} \times_r S^2$ with $e > m > 0$. Then $(M, \mathrm{d}s^2)$ is a four-dimensional Lorentz manifold. The Weyl tensor vanishes on the hypersurface $N := \left\{\left(t, r = \frac{e^2}{m}, \theta, \phi\right) | t \in \mathbb{R}, (\theta, \phi) \in S^2\right\}$, but on $M_{\mathcal{E}} = M - N$ the Weyl tensor is non-zero, in particular $\mathcal{E} = \{0\}$ on $M_{\mathcal{E}}$.

4. Some facts about $\mathcal{E}$

Lemma 3. *Suppose (M, g) is a semi-Riemannian manifold with a harmonic Weyl tensor: $\delta W = 0$. Then $\mathcal{E}$ is an integrable distribution on $M_{\mathcal{E}}$.*

Proof. We first remark that $\delta W = 0$ is equivalent to $\mathrm{d}^\nabla W = 0$ (cf. [1]; 16.41) where $\mathrm{d}^\nabla W$ is given by

$$\mathrm{d}^\nabla W(X, Y, Z, ., .) := (\nabla_X W)(Y, Z, ., .) + (\nabla_Y W)(Z, X, ., .) + (\nabla_Z W)(X, Y, ., .).$$

Suppose U and V are vector fields with values in $\mathcal{E}$. We have to show that $[U, V]$ is a section in $\mathcal{E}$. Since ∇ is torsion free and $\nabla_X W$ has the symmetries of a curvature operator, we obtain for all vector fields X, Y, Z and all $U, V \in \Gamma(\mathcal{E})$:

$$\begin{aligned} W([U, V], X, Y, Z) &= W(\nabla_U V - \nabla_V U, X, Y, Z) \\ &= (\nabla_U W)(V, X, Y, Z) - (\nabla_V W)(U, X, Y, Z) \\ &= \mathrm{d}^\nabla W(U, V, X, Y, Z) - (\nabla_X W)(U, V, Y, Z). \end{aligned}$$

We assumed $\mathrm{d}^\nabla W = 0$ and since $U, V \in \Gamma(\mathcal{E})$ yields

$$(\nabla_X W)(U, V, Y, Z) = 0,$$

$W([U, V], X, Y, Z)$ vanishes for all X, Y, Z which proves $[U, V] \in \Gamma(\mathcal{E})$.

Lemma 4. *Suppose (M^n, g) is a semi-Riemannian manifold and $\mathcal{E}$ is non-degenerate on $M_{\mathcal{E}}$, i.e. the restriction of the metric g to $\mathcal{E}$ is non-degenerate. Then the rank of $\mathcal{E}$ is less or* *equal to* $n-4$ *on the components of* $M_{\mathcal{E}}$ *which are not conformally flat. In particular, if* $\mathcal{E}$ *is non-degenerate and* $\mathrm{rk}(\mathcal{E}) > n-4$ *on* $M_{\mathcal{E}}$*, we conclude* $M_{\mathcal{E}} = M$ *and* g *is conformally flat.*

Proof. $\mathcal{E}$ is non-degenerate if and only if the g-orthogonal complement $\mathcal{E}^{\perp}$ is non-degenerate (cf. [17]). Let $g_|$ and $W_|$ be the restrictions of g, respectively, W to $\mathcal{E}^{\perp}$, then $W_|$ is a Weyl curvature operator for $(\mathcal{E}^{\perp}, g_|)$. Since the space of Weyl curvature operators is trivial in dimension $m \leq 3$, we conclude that $W_| = 0$ if $\mathrm{rk}(\mathcal{E}^{\perp}) \leq 3$. In particular, the Weyl tensor W has to vanish on each component of $M_{\mathcal{E}}$ which has $\mathrm{rk}(\mathcal{E}^{\perp}) \leq 3$.

Corollary 5. *Let* (M, g) *be a four-dimensional Riemannian manifold. Suppose* U *is a connected component of* $M_{\mathcal{E}}$*, then either* (U, g) *is conformally flat or the rank of the bundle* $\mathcal{E} \to U$ *is zero. Moreover, choosing* $h = g$*, the Ricci contraction of* $\mathcal{W}^2 = \mathcal{W}^{\mathrm{t}}\mathcal{W}$ satisfies (*cf.* [1]; *16.75 resp.* [5])

$$\mathfrak{w} = \frac{1}{2}\mathrm{tr}(\mathcal{W}^2)\mathrm{Id}.$$

If (M, g) *is a four-dimensional Lorentz manifold and* U *is a connected component of* $M_{\mathcal{E}}$*, one of the following three cases occurs:*

(1) (U, g) *is conformally flat.*
(2) (U, g) *is of Petrov type N and* $\mathcal{E}$ *is one-dimensional as well as light like.*
(3) $\mathrm{rk}(\mathcal{E}) = 0$ *on* U.

Proof. We only consider the Lorentz case. The above lemma shows that $\mathcal{E}$ cannot be non-degenerate in dimension four, i.e. we have $\mathcal{E} = \{0\}$, $\mathcal{E} = TM$ or $\mathcal{E}$ is light like. Moreover, using the Hodge star operator on $\Lambda^2 M$ and the fact $* \circ \mathcal{W} = \mathcal{W} \circ *$, then $\mathcal{E}$ must be one-dimensional and $\mathrm{Im}(\mathcal{W})$ has to be two-dimensional. This shows that (U, g) is of Petrov type N.

A semi-Riemannian manifold (M, g) is said to be *conformally symmetric* if the Weyl tensor is parallel: $\nabla W = 0$. In particular, a conformally symmetric space has a harmonic Weyl tensor.

Proposition 6. *Suppose* (M^n, g) *is a connected conformally symmetric space. Then* $\mathcal{E}$ *is an integrable distribution on* M *(i.e.* $M_{\mathcal{E}} = M$*) , which is preserved by the Levi–Civita connection:* $\nabla_X U \in \Gamma(\mathcal{E})$ *for all* $U \in \Gamma(\mathcal{E})$ *and* $X \in \Gamma(TM)$.

Moreover, if $\mathcal{E}$ *is non-degenerate of rank* $m \leq n-4$, (M, g) *is locally a Riemannian product:*

$$M \overset{\mathrm{loc}}{=} M_1 \times M_2, \qquad g \overset{\mathrm{loc}}{=} \pi_1^*(g_1) + \pi_2^*(g_2)$$

where $\pi_j : M_1 \times M_2 \to M_j$, $j = 1, 2$, *are the projections and* (M_j, g_j) *are semi-Riemannian manifolds with* $\pi_1^*(TM_1) = \mathcal{E}$ *and* $\pi_2^*(TM_2) = \mathcal{E}^{\perp}$.

Proof. Let p and q be arbitrary points in M and $P_\gamma : T_pM \to T_qM$ be the parallel transport for the smooth curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. P_γ is an isometric isomorphism. The Levi–Civita connection on $T^*M \otimes T^*M \otimes T^*M$ induces another parallel transport $\overline{P}_\gamma : (T_p^*M)^3 \to (T_q^*M)^3$. Since the Weyl tensor is parallel, we obtain

$$\overline{P}_\gamma(W(v, ., ., .)) = W(P_\gamma v, ., ., .) \tag{8}$$

for any $v \in T_pM$. In order to see this let $\theta = \theta(t)$ be the parallel transport of $\theta(0) := W(v, ., ., .)$ along γ. Moreover, suppose $X = X(t)$ is the parallel transport of v along γ and define $\eta(t) := W(X(t), ., ., .)$. By definition we have $\nabla_{\gamma'}\theta = 0$ and $\nabla_{\gamma'}X = 0$, i.e. we conclude from $\nabla W = 0$

$$\nabla_{\gamma'}\eta = (\nabla_{\gamma'}W)(X, ., ., .) + W(\nabla_{\gamma'}X, ., ., .) = 0.$$

Since parallel transport is unique, $\eta(0) = \theta(0)$ supplies $\eta(1) = \theta(1)$ and this shows equation (8). Thus, if v is contained in $\mathcal{E}_p$, $W(v, ., ., .) = 0$ and equation (8) yields $W(P_\gamma v, ., ., .) = 0$ which is equivalent to $P_\gamma v \in \mathcal{E}_q$. This proves $P_\gamma : \mathcal{E}_p \to \mathcal{E}_q$ and since P_γ is injective, we conclude $\dim \mathcal{E}_p \le \dim \mathcal{E}_q$. Considering the parallel transport from q to p yields $\dim \mathcal{E}_q \le \dim \mathcal{E}_p$. Therefore, $\dim \mathcal{E}_p$ does not depend on the choice of $p \in M$ and $\mathcal{E}$ is a subbundle of TM. Furthermore, we have already proved that $\mathcal{E}$ is integrable if $\delta W = 0$.

Let U be a vector field with values in $\mathcal{E}$, then $W(\nabla_X U, Y, Z, T) = 0$ (use $\nabla W = 0$) shows $\nabla_X U \in \Gamma(\mathcal{E})$. If $\mathcal{E}$ is non-degenerate, $\mathcal{E}^\perp$ is non-degenerate and TM decomposes orthogonal into $\mathcal{E} \oplus \mathcal{E}^\perp$. Since parallel transport preserves the decomposition $TM = \mathcal{E} \oplus \mathcal{E}^\perp$, we obtain the second claim from holonomy theory.

Remark 7. If (M, g) is an irreducible locally symmetric space, we conclude $M_\mathcal{E} = M$ and either g is of constant sectional curvature or $\mathcal{E} = \{0\}$.

5. Conformal *C*-spaces

A semi-Riemannian manifold (M^n, g), $n \ge 4$, is conformally related to a space with harmonic Weyl tensor if and only if there is a function ϕ with:

$$\delta W_Z = (n - 3)\mathcal{W}(Z^* \wedge \mathrm{d}\phi) \tag{9}$$

[cf. equation (4)]. In this case δW_Z is the two form defined by

$$\delta W_Z(X, Y) := \delta W(X, Y, Z).$$

A necessary condition for a solution of (9) is $\delta W_Z \in \mathrm{Im}(\mathcal{W})$, but we compute $\mathrm{d}\phi$ explicitly which means that this condition will be superfluous. Let h be a Riemannian metric on TM, $\mathcal{W}^{\mathrm{t}}$ be the adjoint of $\mathcal{W}$ with respect to h, $\mathfrak{w}$ be the negative Ricci contraction of $\mathcal{W}^{\mathrm{t}}\mathcal{W}$ and $\mathfrak{w}^\#$ be the Moore–Penrose inverse of $\mathfrak{w}$ which is well defined and smooth on $M_\mathcal{E}$. Applying

$\mathcal{W}^{\mathrm{t}}$ to equation (9) yields

$$\mathcal{W}^{\mathrm{t}}(\delta W_Z) = (n-3)\mathcal{W}^{\mathrm{t}}\mathcal{W}(Z^* \wedge \mathrm{d}\phi).$$

Moreover, suppose $e_1, \ldots, e_n$ is a g-orthonormal base of T_pM, i.e. $g(e_i, e_j) = \epsilon_i\delta_{ij}$ with $\epsilon_i := g(e_i, e_i) = \pm 1$. Let $\eta_1, \ldots, \eta_n$ be the corresponding cobase in T_p^*M, then $\eta_j := \epsilon_j e_j^*$ shows

$$\sum_{i=1}^{n} \epsilon_i e_i \llcorner \mathcal{W}^{\mathrm{t}}(\delta W_{e_i}) = (n-3)\sum_{i=1}^{n} \epsilon_i e_i \llcorner \mathcal{W}^{\mathrm{t}}\mathcal{W}(e_i^* \wedge \mathrm{d}\phi) = (n-3)\mathfrak{w}(\mathrm{d}\phi).$$

Thus, the h-orthogonal projection of $\mathrm{d}\phi$ to $(\mathcal{E}^*)^{\perp h}$ is given by

$$\mathfrak{w}^{\#}\mathfrak{w}(\mathrm{d}\phi) = \frac{1}{n-3}\mathfrak{w}^{\#}\left(\sum_{i=1}^{n} \epsilon_i e_i \llcorner \mathcal{W}^{\mathrm{t}}(\delta W_{e_i})\right).$$

In particular, we obtain

$$\mathrm{d}\phi = \frac{1}{n-3}\mathfrak{w}^{-1}\left(\sum_{i=1}^{n} \epsilon_i e_i \llcorner \mathcal{W}^{\mathrm{t}}(\delta W_{e_i})\right)$$

on the components of $M_{\mathcal{E}}$ where $\mathrm{rk}(\mathcal{E}) = 0$.

Definition 8. Let (M^n, g) be a smooth semi-Riemannian manifold of dimension $n \geq 4$ and $e_1, \ldots, e_n$ be a g-orthonormal base with $\epsilon_j := g(e_j, e_j) \in \{\pm 1\}$. For any choice of a smooth Riemannian metric h on TM the vector field $\mathbb{T}$ given by

$$\mathbb{T}^* = g(\mathbb{T}, .) := \frac{1}{n-3}\mathfrak{w}^{\#}\sum_{i=1}^{n} \epsilon_i e_i \llcorner \mathcal{W}^{\mathrm{t}}(\delta W_{e_i}) = \frac{1}{2(n-3)}\mathfrak{w}^{\#}\sum_{i,j,k}(\mathcal{W}^{\mathrm{t}})_{ijk}\,\delta W^{ijk} \quad (10)$$

and the (0, 3) tensor

$$C_{\mathbb{T}} := \mathrm{d}^{\nabla}\mathfrak{k} - W(., ., ., \mathbb{T})$$

are smooth on the open and dense subset $M_{\mathcal{E}} \subseteq M$.

In the definition of $C_{\mathbb{T}}$ we used $\mathrm{d}^{\nabla}\mathfrak{k}$ instead of δW in order to get the Cotton tensor in dimension $n = 3$, but for the computations we consider:

$$C_{\mathbb{T}} = \frac{1}{n-3}\delta W - W(., ., ., \mathbb{T}). \quad (11)$$

In general, the vector field $\mathbb{T}$ as well as the tensor $C_{\mathbb{T}}$ depend on the choice of the Riemannian metric h. If g is indefinite, there is no canonical choice of h, hence it seems difficult to get a tensor $C_{\mathbb{T}}$ only depending on g. However, if $C_{\mathbb{T}}$ vanishes, $\mathbb{T}$ is unique in $TM/\mathcal{E}$:

Lemma 9. *Suppose* (M, g) *is a semi-Riemannian manifold and* $\mathbb{T}_1$ *as well as* $\mathbb{T}_2$ *are the vector fields defined in (10) with respect to Riemannian metrics* h_1 *and* h_2. *Then* $C_{\mathbb{T}_1}$ *vanishes if and only if* $C_{\mathbb{T}_2}$ *vanishes. Moreover, we have* $C_{\mathbb{T}_1} = C_{\mathbb{T}_2}$ *if and only if* $\mathbb{T}_1 - \mathbb{T}_2 \in \Gamma(\mathcal{E})$.

Proof. The second claim is obvious. In order to see the first claim, we use the definition of $\mathbb{T}_2$ and the definition of $C_{\mathbb{T}_1}$ in equation (11):

$$\mathbb{T}_2^* = \mathfrak{w}_2^\# \left(\sum_{i=1}^n \epsilon_i e_i \llcorner \mathcal{W}^{\mathrm{t}}(C_{\mathbb{T}_1}(.,.,e_i) + \mathcal{W}(e_i^* \wedge \mathbb{T}_1^*)) \right)$$
$$= \mathfrak{w}_2^\# \mathfrak{w}_2(\mathbb{T}_1^*) + \mathfrak{w}_2^\# \left(\sum_{i=1}^n \epsilon_i e_i \llcorner \mathcal{W}^{\mathrm{t}}(C_{\mathbb{T}_1}(.,.,e_i)) \right).$$

Thus, $C_{\mathbb{T}_1} = 0$ and $\mathfrak{w}_2^\#\mathfrak{w}_2(\mathbb{T}_2^*) = \mathbb{T}_2^*$ imply $\mathfrak{w}_2^\#\mathfrak{w}_2(\mathbb{T}_1^* - \mathbb{T}_2^*) = 0$. Since $\mathfrak{w}_2^\#\mathfrak{w}_2$ is the orthogonal projection $T^*M \to (\mathcal{E}^*)^{\perp h_2}$ with respect to h_2, we conclude $\mathbb{T}_1^* - \mathbb{T}_2^* \in \Gamma(\mathcal{E}^*)$. Therefore, $\mathbb{T}_1 - \mathbb{T}_2 \in \Gamma(\mathcal{E})$ and $C_{\mathbb{T}_1} = 0$ yield $C_{\mathbb{T}_2} = 0$. □

Remark 10. The vector field $\mathbb{T}$ does not depend on a conformal transformation of the Riemannian metric h.

Proposition 11. *Suppose* $\bar{g} = \psi^{-2} g$ *is a conformal transformation with* $\psi = e^\phi$, *then* $\bar{C}_{\bar{\mathbb{T}}}$ *equals* $C_{\mathbb{T}}$, *while for the definition of* $\bar{\mathbb{T}}$ *and* $\mathbb{T}$ *the Riemannian metric h is fixed or scaled by a conformal factor.*

Proof. $\bar{W} = \psi^{-2} W$ supplies $\bar{\mathcal{W}} = \psi^2 \mathcal{W}$ and $\bar{\mathcal{W}}^{\mathrm{t}} = \psi^2 \mathcal{W}^{\mathrm{t}}$, in particular, we obtain $\bar{\mathfrak{w}} = \psi^4 \mathfrak{w}$ and $\bar{\mathfrak{w}}^\# = \psi^{-4}\mathfrak{w}^\#$. If $e_1, \ldots, e_n$ is an orthonormal base with respect to g, $\overline{e_1} := \psi e_1, \ldots, \overline{e_n} := \psi e_n$ is an orthonormal base with respect to $\bar{g}$, i.e.

$$\bar{\bar{\mathbb{T}}}^* = \frac{1}{n-3} \bar{\mathfrak{w}}^\# \left(\sum_{i=1}^n \epsilon_i \overline{e_i} \llcorner \bar{\mathcal{W}}^{\mathrm{t}}(\bar{\delta} \bar{W}_{\overline{e_i}}) \right)$$
$$= \frac{1}{n-3} \mathfrak{w}^\# \left(\sum_{i=1}^n \epsilon_i e_i \llcorner \mathcal{W}^{\mathrm{t}}(\delta W_{e_i} - (n-3)\mathcal{W}(e_i^* \wedge \mathrm{d}\phi)) \right) = \mathbb{T}^* - \mathfrak{w}^\# \mathfrak{w}(\mathrm{d}\phi).$$

Thus, the vector fields satisfy:

$$\bar{\mathbb{T}} = \psi^2 \mathbb{T} - \psi^2 (\nabla^g \phi)_{\mathcal{E}^{\perp h^*}}, \tag{12}$$

where $(\nabla^g \phi)_{\mathcal{E}^{\perp h^*}}$ is the projection of $\nabla\phi$ to the h^*-orthogonal complement of $\mathcal{E}$ in TM. Since $W(.,.,.,(\nabla\phi)_{\mathcal{E}})$ vanishes, we conclude the conformal invariance of the tensor $C_{\mathbb{T}}$ if $n > 3$ (use (4)):

$$\bar{C}_{\bar{\mathbb{T}}} = \frac{1}{n-3} \bar{\delta} \bar{W} - \bar{W}(.,.,., \bar{\mathbb{T}})$$
$$= \frac{1}{n-3} \delta W - W(.,.,., \nabla^g \phi) - W(.,.,., \mathbb{T} - (\nabla^g \phi)_{\mathcal{E}^{\perp h^*}}) = C_{\mathbb{T}}.$$

In dimension $n = 3$ the conformal invariance of the Cotton tensor $C_{\mathbb{T}} = \mathrm{d}^{\nabla}\mathfrak{k}$ is already known. □

Since $C_{\mathbb{T}}$ is trivial on a space with harmonic Weyl tensor, the vanishing of $C_{\mathbb{T}}$ is a necessary condition for a semi-Riemannian manifold to be locally conformally related to a C-space. However, $C_{\mathbb{T}} = 0$ is a sufficient condition if and only if there is a section V in $\mathcal{E}$ in such a way that $\mathbb{T} + V$ is a (differentiable) gradient field on M. Hence, we conclude the following in case $\mathrm{rk}(\mathcal{E}) = 0$.

Proposition 12. *Let (M^n, g) be a simply connected semi-Riemannian manifold of dimension $n \geq 4$ such that $\mathrm{rk}(\mathcal{E}) = 0$ on $M_{\mathcal{E}}$. Then (M, g) is (globally) conformally related to a space with harmonic Weyl tensor if and only if $\mathrm{d}\mathbb{T}^*$ and $C_{\mathbb{T}}$ vanish on $M_{\mathcal{E}}$ as well as $\mathbb{T}$ can be (differentiable) extended to M. Moreover, if $\det \mathcal{W}$ does not vanish on $M_{\mathcal{E}}$, the condition $\mathrm{d}\mathbb{T}^* = 0$ follows from $C_{\mathbb{T}} = 0$.*

Proof. The claims follow from the uniqueness of the conformal factor (up to scaling) and the definition of $\mathbb{T}$ and $C_{\mathbb{T}}$. Since M is simply connected and $\mathbb{T}^*$ is exact, there is a function $\phi : M \to \mathbb{R}$ with $\nabla^g\phi = \mathbb{T}$. Set $\psi := e^{\phi}$, then $\psi^{-2}g$ is a space with harmonic Weyl tensor. In order to see the last claim we consider the divergence of $C_{\mathbb{T}}$ with respect to the third argument. A straightforward calculation shows (cf. [15])

$$\delta_3(C_{\mathbb{T}}) = C_{\mathbb{T}}(.,.,\mathbb{T}) - \mathcal{W}(\mathrm{d}\mathbb{T}^*),$$

i.e. we conclude $\mathrm{d}\mathbb{T}^* = 0$ from the injectivity of $\mathcal{W}$ and $C_{\mathbb{T}} = 0$. □

6. Conformal Einstein spaces

A semi-Riemannian manifold (M, g) of dimension $n \geq 3$ is called *Einstein space* if the traceless Ricci tensor $\mathrm{Ric}^{\circ} := \mathrm{Ric} - \frac{\mathrm{scal}}{n} g$ vanishes. (M, g) is said to be a *conformal Einstein space* if g is locally conformally related to an Einstein space. Let $(M, g) \to (M, \overline{g} := \psi^{-2}g)$ be a conformal transformation with $\psi = \mathrm{e}^{\phi}$. The Ricci tensor has the following transformation behavior (cf. [13]; Lemma A.1 or [1]):

$$\overline{\mathrm{Ric}} = \mathrm{Ric} + (n-2)[\nabla^2\phi + \mathrm{d}\phi \otimes \mathrm{d}\phi] + [\triangle\phi - (n-2)\langle\nabla^g\phi, \nabla^g\phi\rangle]g, \tag{13}$$

where $\nabla^2\phi$ is the Hessian of ϕ (i.e. $\nabla^2\phi(X, Y) = \langle\nabla_X\nabla^g\phi, Y\rangle$) and $\triangle\phi$ is the trace of $\nabla^2\phi$. If $(M, \overline{g})$ is an Einstein space, we conclude for the Ricci tensor of (M, g):

$$0 = \mathrm{Ric}^{\circ} + (n-2)[\nabla^2\phi + \mathrm{d}\phi \otimes \mathrm{d}\phi] - \frac{n-2}{n}[\triangle\phi + \left|\nabla^g\phi\right|^2]g. \tag{14}$$

Moreover, suppose (M, g) is a semi-Riemannian manifold and $\phi : M \to \mathbb{R}$ is a function which satisfies (14), then $(M, \mathrm{e}^{-2\phi}g)$ is an Einstein space [use (13)].

Definition 13. Let V be a vector field, then the traceless (0, 2) tensor field:

$$E_V := \mathrm{Ric}^\circ + (n-2)\left(\nabla V^* + V^* \otimes V^* - \frac{1}{n}[\mathrm{div}(V) + g(V, V)]g\right)$$

is called *conformal Ricci tensor with respect to V*.

The tensor field E_V is symmetric if and only if ∇V^* is symmetric, i.e. if and only if V^* is closed. Thus, a semi-Riemannian manifold (M, g) is locally conformally related to an Einstein space if and only if there is a vector field V with $E_V = 0$. If M is simply connected and E_V vanishes, there is a function $\psi = \mathrm{e}^\phi$ which gives the Einstein space $(M, \psi^{-2}g)$. In this case V equals $\nabla^g\phi$. The differential Bianchi identity shows that an Einstein space has a harmonic Weyl tensor [cf. (3)]. Thus, the first candidate of a vector field V which satisfies $E_V = 0$ is the vector field $\mathbb{T}$ given in (10).

Lemma 14. *Let $\overline{g} = \psi^{-2}g$, $\psi = e^\phi$, be a conformal transformation and $\overline{\mathbb{T}}$ as well as $\mathbb{T}$ be the corresponding vector fields defined in (10) with respect to a fixed Riemannian metric h or Riemannian metrics $\overline{h}$ and h with $\overline{h} = \psi^{-2}h$. If $\mathrm{d}\phi$ is contained in the h-orthogonal complement of $\mathcal{E}^* \subseteq T^*M$ in every point of $M_{\mathcal{E}}$, the conformal Ricci tensors satisfy:*

$$\overline{E}_{\overline{\mathbb{T}}} = E_{\mathbb{T}}.$$

Proof. Introduce the (0, 2) tensor

$$F_V := \nabla V^* + V^* \otimes V^* - \frac{1}{n}[\mathrm{div}(V) + g(V, V)]g.$$

One easily verifies for two vector fields V and Z:

$$F_{V+Z} = F_V + F_Z + V^* \otimes Z^* + Z^* \otimes V^* - \frac{2}{n}g(V, Z)g. \tag{15}$$

Moreover, using (2) a straightforward calculation yields:

$$\overline{F}_{\psi^2 V} = F_V + \mathrm{d}\phi \otimes V^* + V^* \otimes \mathrm{d}\phi - \frac{2}{n}g(\nabla^g\phi, V)g.$$

Since ψ^{-2} is the conformal factor, equation (12) supplies $\overline{\mathbb{T}} = \psi^2(\mathbb{T} - Y)$ where Y is determined by the h-orthogonal decomposition of $\mathrm{d}\phi = X^* + Y^*$ in $\mathcal{E}^* \oplus (\mathcal{E}^*)^{\perp h} = T^*M$ [$X^* : M_{\mathcal{E}} \to \mathcal{E}^*$ and $Y^* : M_{\mathcal{E}} \to (\mathcal{E}^*)^{\perp h}$]. Thus, we obtain:

$$\begin{aligned}\overline{F}_{\overline{\mathbb{T}}} &= F_{\mathbb{T}-Y} + \mathrm{d}\phi \otimes (\mathbb{T}^* - Y^*) + (\mathbb{T}^* - Y^*) \otimes \mathrm{d}\phi - \frac{2}{n}g(\nabla^g\phi, \mathbb{T} - Y)g \\ &= F_{\mathbb{T}} - F_Y + X^* \otimes (\mathbb{T}^* - Y^*) + (\mathbb{T}^* - Y^*) \otimes X^* - \frac{2}{n}g(X, \mathbb{T} - Y)g.\end{aligned}$$

Use equation (13) to conclude that the traceless Ricci tensors are related by $(n-2)F_{\nabla\phi}$. Hence, the definition of E_V and $\nabla^g\phi = X + Y$ show:

$$\begin{aligned}\bar{E}_{\bar{\mathbb{T}}} &= \overline{\mathrm{Ric}}^{\circ} + (n-2)\bar{F}_{\bar{\mathbb{T}}}\\ &= \mathrm{Ric}^{\circ} + (n-2)\Big[F_{X+Y} + F_{\mathbb{T}} - F_Y + X^* \otimes (\mathbb{T}^* - Y^*)\\ &\quad + (\mathbb{T}^* - Y^*) \otimes X^* - \frac{2}{n} g(X, \mathbb{T} - Y)\Big]\\ &= E_{\mathbb{T}} + (n-2)\left[F_X + X^* \otimes \mathbb{T}^* + \mathbb{T}^* \otimes X^* - \frac{2}{n} g(X, \mathbb{T}) g\right].\end{aligned} \tag{16}$$

But we assumed that $\mathrm{d}\phi$ takes it values in $(\mathcal{E}^*)^{\perp h} \subseteq T^*M$, i.e. $X = 0$ supplies the claim. □

Remark 15. We have proved that $E_{\mathbb{T}}$ is a conformal invariant for all semi-Riemannian manifolds (M, g) with $\mathrm{rk}(\mathcal{E}) = 0$. In particular, if (M, g) is a Riemannian four manifold, $E_{\mathbb{T}}$ is conformally invariant on the open subset $\{p \in M | W(p) \neq 0\} \subseteq M$.

Theorem 16. *Suppose (M, g) is a simply connected semi-Riemannian manifold of dimension $n \geq 4$ and with $\mathrm{rk}(\mathcal{E}) = 0$ on $M_{\mathcal{E}}$. Then (M, g) is (globally) conformally related to an Einstein space if and only if $E_{\mathbb{T}}$ vanishes on $M_{\mathcal{E}}$ and $\mathbb{T}$ is extendible to a vector field on M. In particular, this equivalence does not depend on the choice of the Riemannian metric h in order to define $\mathbb{T}$.*

Proof. Since $E_{\mathbb{T}}$ is a conformal invariant on manifolds with $\mathrm{rk}(\mathcal{E}) = 0$ and $\mathbb{T}$ vanishes on Einstein spaces, $E_{\mathbb{T}} = 0$ is a necessary condition. Conversely, $E_{\mathbb{T}} = 0$ implies that $\mathbb{T}^*$ is closed. Since M is simply connected, there is a function $\phi : M \to \mathbb{R}$ with $\mathbb{T} = \nabla\phi$, and the above computations show that $\mathrm{e}^{-2\phi} g$ is an Einstein metric on M. □

The following corollary was already proved in [15] and it is a result of the main theorem and the analytic regularity of Einstein metrics (cf. [6]).

Corollary 17. *Suppose (M, g) is a connected four-dimensional Riemannian manifold. Then (M, g) is a conformal Einstein space if and only if g is conformally flat or $E_{\mathbb{T}}$ vanishes and $\mathbb{T}$ is extendible to a vector field on M.*

$E_{\mathbb{T}}$ is a conformal invariant in the category of metrics with $\mathrm{rk}(\mathcal{E}) = 0$ while $C_{\mathbb{T}}$ is a conformal invariant for any metric. In particular, the vanishing of $E_{\mathbb{T}}$ is only sufficient for a metric to be conformally Einstein, in case $\mathrm{rk}(\mathcal{E}) > 0$ its vanishing is not necessary. Conversely, $C_{\mathbb{T}} = 0$ is a necessary condition for all conformal Einstein spaces but in case of dimension $n \geq 4$ it is not sufficient. There is another invariant which has to vanish for a conformal Einstein space. The *generalized Bach tensor* $B_{\mathbb{T}}$ (cf. [14]) is conformally invariant. Since $B_{\mathbb{T}}$ vanishes for Einstein spaces, $B_{\mathbb{T}} = 0$ is necessary for g to be conformally Einstein but according to [12,18,16] it is not sufficient to guarantee that g is conformally

Einstein. In the last section, we consider the case $\mathrm{rk}(\mathcal{E}) = 1$, but if $\mathrm{rk}(\mathcal{E})$ is greater than one, the problem of giving tensorial conditions which are necessary and sufficient for a conformal Einstein space remains unsolved.

7. The case $\mathrm{rk}(\mathcal{E}) = 1$

Suppose (M, g) is a semi-Riemannian manifold with $\mathrm{rk}(\mathcal{E}) = 1$. We conclude from the above considerations that if $(M, \mathrm{e}^{-2\phi} g)$ is an Einstein space, the gradient of ϕ is given by the vector field:

$$S := \mathbb{T} + fV$$

where $f : M \to \mathbb{R}$ is a smooth function and V is a fixed nowhere vanishing vector field with values in $\mathcal{E}$. Since S is a gradient field, we obtain:

$$0 = \mathrm{d}S^* = \mathrm{d}\mathbb{T}^* + \mathrm{d}f \wedge V^* + f\mathrm{d}V^*. \tag{17}$$

If $\mathrm{d}\mathbb{T}^*$ is not a section in $\mathcal{E}^* \wedge T^*M$, this yields an unique obstruction on f and therefore a solution of the problem. Thus, we can suppose that $\mathrm{d}\mathbb{T}^*$ and $\mathrm{d}V^*$ are sections in $\mathcal{E}^* \wedge T^*M$.

Definition 18. *In the case that $\mathcal{E}$ is not light like, we normalize $V \in \Gamma(\mathcal{E})$ in such a way that $\epsilon := |V|^2 = \pm 1$ and we set $V_c := \epsilon V$. If $\mathcal{E}$ is light like, we choose $V \in \Gamma(\mathcal{E})$ such that there is another light like vector V_c which defines a non-degenerate rank two distribution $\mathcal{E} \oplus \mathbb{R} \cdot V_c$ (i.e. the restriction of g to this distribution is non-degenerate). We normalize again by $g(V, V_c) = 1$.*

Equation (17) yields:

$$0 = \mathrm{d}f(V_c)V^* - \mathrm{d}f + fV_c \llcorner \mathrm{d}V^* + V_c \llcorner \mathrm{d}\mathbb{T}^*. \tag{18}$$

We want to compute f assuming $E_S = 0$. We conclude from (15):

$$\begin{aligned} E_S = E_{\mathbb{T}} &+ (n-2)\left[\mathrm{d}f \otimes V^* - \frac{1}{n}\mathrm{d}f(V)g + f^2 V^* \otimes V^* - \frac{1}{n} f^2 |V|^2 g\right] \\ &+ (n-2)f\left[\nabla V^* - \frac{1}{n}\mathrm{div}(V)g + V^* \otimes \mathbb{T}^* + \mathbb{T}^* \otimes V^* - \frac{2}{n}\langle V, \mathbb{T}\rangle g\right]. \end{aligned}$$

Therefore, equation (18) supplies

$$E_S = E_{\mathbb{T}} + (n-2)\left[D + fA + (\mathrm{d}f(V_c) + f^2)\left(V^* \otimes V^* - \frac{1}{n}|V|^2 g\right)\right] \tag{19}$$

where A and D are given as follows

$$\begin{aligned} A &:= \nabla V^* - \tfrac{1}{n}\mathrm{div}(V)g + V^* \otimes \mathbb{T}^* + \mathbb{T}^* \otimes V^* - \tfrac{2}{n}\langle V, \mathbb{T}\rangle g \\ &\quad + V_c \llcorner \mathrm{d}V^* \otimes V^* - \tfrac{1}{n}\mathrm{d}V^*(V_c, V)g \\ D &:= V_c \llcorner \mathrm{d}\mathbb{T}^* \otimes V^* - \tfrac{1}{n}\mathrm{d}\mathbb{T}(V_c, V)g. \end{aligned}$$

Denote by $\mathcal{U}$ the rank one subbundle of $T^*M \otimes T^*M$ which is generated by $V^* \otimes V^* - \frac{1}{n}|V|^2 g$. If $E_{\mathbb{T}} + (n-2)D$ is nowhere a section in $\mathcal{U}$, the condition $E_S = 0$ supplies an unique obstruction on f and we had solved the problem. Thus, it remains to consider the case that $E_{\mathbb{T}} + (n-2)D$ and A are sections in $\mathcal{U}$. We apply the vector field V_c in both arguments to (19), this leads to:

$$0 = E_S(V_c, V_c) = b_1 + fb_2 + \kappa(\mathrm{d}f(V_c) + f^2)$$

where $b_1 := E_{\mathbb{T}}(V_c, V_c)$, $b_2 := (n-2)A(V_c, V_c)$ and $\kappa := (n-2)\left(1 - \frac{1}{n}|V|^4\right)$. We insert $\mathrm{d}f(V_c)$ into (18) and obtain a first order system of Riccati type:

$$0 = \mathrm{d}f + f^2 V^* + fY^* + Z^* \tag{20}$$

with

$$Y^* := \frac{b_2}{\kappa} V^* - V_c \llcorner \mathrm{d}V^*, \qquad Z^* := \frac{b_1}{\kappa} V^* - V_c \llcorner \mathrm{d}\mathbb{T}^*.$$

Suppose f is a solution of this system, then (20) yields equation (18) and furthermore, $\mathrm{d}f(V_c) + f^2 = -fb_2/\kappa - b_1/\kappa$ simplifies (19) to $E_S = Q + fP$:

$$Q := E_{\mathbb{T}} + (n-2)D - \tfrac{E_{\mathbb{T}}(V_c, V_c)}{1 - \frac{1}{n}|V|^4}\left(V^* \otimes V^* - \tfrac{1}{n}|V|^2 g\right)$$

$$P := (n-2)\left[A - \tfrac{A(V_c, V_c)}{1 - \frac{1}{n}|V|^4}\left(V^* \otimes V^* - \tfrac{1}{n}|V|^2 g\right)\right].$$

In particular, if P does not vanish, we obtain an unique obstruction on f. To be more precise, let h be a Riemannian metric on $T^*M \otimes T^*M$, then $0 = h(E_S, P) = h(Q, P) + f \cdot h(P, P)$ yields $f = -h(Q, P)/h(P, P)$.

Proposition 19. *Let (M, g) be a semi-Riemannian manifold with $\mathrm{rk}(\mathcal{E}) = 1$ and let h be an arbitrary Riemannian metric on $T^*M \otimes T^*M$. Then (M, g) is locally conformally related to an Einstein space if and only if*

- $P \neq 0$ *(in a neighborhood): The vector field*

$$S := \mathbb{T} - \frac{h(Q, P)}{h(P, P)} V$$

  satisfies $E_S = 0$.
- $P = 0$ *(in a neighborhood): The system* (20) *has a solution f and $Q = 0$.*

We will not discuss the necessary and sufficient integrability conditions for (20) in detail. If we take the exterior derivative of (20) and insert $\mathrm{d}f$, this leads to

$$f^2(\mathrm{d}V^* + V^* \wedge Y^*) + f(\mathrm{d}Y^* + 2V^* \wedge Z^*) + \mathrm{d}Z^* + Y^* \wedge Z^* = 0.$$

Since $\mathrm{d}V^* + V^* \wedge Y^* = 0$ is equivalent to P is symmetric (cf. proof of Proposition 20), the condition $P = 0$ yields an unique obstruction on f as long as $\mathrm{d}Y^* + 2V^* \wedge Z^*$ is not zero. Obviously, $\mathrm{d}Z^* + Y^* \wedge Z^*$ has to vanish if $\mathrm{d}Y^* + 2V^* \wedge Z^* = 0$. In particular, as the discussion of the Bernoulli system below shows, the vanishing of $\mathrm{d}Z^* + Y^* \wedge Z^*$ and $\mathrm{d}Y^* + 2V^* \wedge Z^*$ are supposed to be sufficient for the existence of a solution of (20).

This method can also provide non-trivial solutions of

$$\nabla^2\psi = \frac{\triangle\psi}{n} g \tag{21}$$

on Einstein manifolds. If (M, g) is an Einstein space and $\overline{g} = \psi^{-2}g$ is a conformal transformation of g, then $\overline{g}$ is an Einstein space if and only if ψ is a solution of (21) (cf. [13]). Suppose now that (M, g) is a simply connected Einstein space with $\mathrm{rk}(\mathcal{E}) = 1$ on M. We use the above approach to compute this solution. If $\overline{g} = \mathrm{e}^{-2\phi}g$ is an Einstein space, $\nabla\phi$ has to be a vector field with values in $\mathcal{E}$ (since g and $\overline{g}$ are C-spaces), in particular $\nabla\phi = fV$ for a function f. Since g is Einstein, we have $\mathbb{T} = 0$ and $E_{\mathbb{T}} = 0$. Hence, the system (20) reduces to a Bernoulli equation:

$$0 = \mathrm{d}f + f^2V^* + fY^*. \tag{22}$$

Take $\mathrm{d}(fV^*) = 0$ into consideration, the necessary and sufficient integrability conditions for this system are

$$\mathrm{d}V^* = V^* \wedge (V_c \llcorner \mathrm{d}V^*) = -V^* \wedge Y^* \quad \text{and} \quad \mathrm{d}Y^* = 0.$$

As usual, we divide the Bernoulli equation by f^2 and obtain a linear system which leads to the solution of (22). Suppose $Y^* = \mathrm{d}\alpha$ for a function α, then the integrability conditions yield that $\mathrm{e}^{-\alpha}V^*$ is exact. Therefore, we can assume $V^* = \mathrm{e}^{\alpha}\mathrm{d}\beta$ for a function β, and a solution of the Bernoulli system is given by

$$f := \frac{1}{\beta \mathrm{e}^{\alpha}}.$$

f is uniquely determined up to the choice of a constant c in $\beta = \beta_0 + c$. Thus, using potential theory leads to the solution of the equation (21) (note that $\nabla\phi = fV$ and $\psi = \mathrm{e}^{\phi}$).

If (M, g) is an Einstein manifold, the fact $\mathbb{T} = 0$ reduces the tensor P to

$$P = (n-2)\Bigg[\nabla V^* + V_c \llcorner \mathrm{d}V^* \otimes V^* - \mu V^* \otimes V^* - \frac{1}{n}(\mathrm{div}(V) + \mathrm{d}V^*(V_c, V) - \mu|V|^2)g\Bigg]$$

where μ is given by $\langle \nabla_{V_c} V, V_c \rangle$ if V is light like and given by $-\frac{1}{n-1}\mathrm{div}(V)|V|^2$ if V is space or time like.

Proposition 20. *Suppose* (M^n, g) *is an Einstein manifold of dimension* $n \geq 4$ *and with* $\mathrm{rk}(\mathcal{E}) = 1$. *Let* V *and* V_c *be the vector fields introduced in* Definition 18. *Then locally there is a non-constant function* ϕ *which gives an Einstein metric* $\mathrm{e}^{-2\phi} g$ *if and only if* $P = 0$ *and* $\mathrm{d}Y^* = 0$, *where* $Y^* = \mu V^* - V_c \llcorner \mathrm{d}V^*$.

Proof. That $P = 0$ and $\mathrm{d}Y^* = 0$ are necessary follows from the above considerations. Suppose $U \subseteq M$ is simply connected and open. Set $Y^* = \mathrm{d}\alpha$ on U. The relation $2\nabla X^* = L_X g + \mathrm{d}X^*$ proves that P (respectively, A) is a symmetric tensor if and only if

$$\mathrm{d}V^* = V^* \otimes V_c \llcorner \mathrm{d}V^* - V_c \llcorner \mathrm{d}V^* \otimes V^* = V^* \wedge V_c \llcorner \mathrm{d}V^*$$

(this is equivalent to the fact that $\mathrm{d}V^*$ is a section in $\mathcal{E}^* \wedge T^*M$). Thus, $P = 0$ yields:

$$\mathrm{d}V^* = V^* \wedge V_c \llcorner \mathrm{d}V^* = -V^* \wedge Y^* = \mathrm{d}\alpha \wedge V^*.$$

In particular, $\mathrm{e}^{-\alpha} V^*$ is exact and equals $\mathrm{d}\beta$ for some function β. Choose U and β in such a way that $\beta \neq 0$ on U. Then $f := \mathrm{e}^{-\alpha}/\beta$ is a solution of the system (22) and $\mathrm{d}(fV^*) = 0$. Moreover, a straightforward calculation shows $\mathrm{d}f(V_c) + f^2 = -\mu f$. Thus, equation (19) reduces to:

$$E_{fV} = (n-2) f \left[A - \mu \left(V^* \otimes V^* - \frac{1}{n} |V|^2 g \right) \right] = fP.$$

where E_{fV} vanishes if and only if $(U, \mathrm{e}^{-2\phi} g)$ is Einstein ($\nabla \phi = fV$). $\square$

The last two propositions are of particular interest in the four-dimensional Lorentz case, since the rank of $\mathcal{E}$ is less or equal to one on the components of $M_{\mathcal{E}}$ which are not conformally flat. If $\overline{g} = \psi^{-2} g$ is a conformal transformation of two four-dimensional Einstein Lorentz spaces which are not conformally flat, then g and $\overline{g}$ have to be Ricci flat as well as of Petrov type N. Moreover, the gradient of the conformal factor ψ is light like and parallel (i.e. $\nabla^2 \psi = 0$). Hence, a four-dimensional Einstein Lorentz manifold (M, g) which is not of constant sectional curvature admits a non-trivial parallel vector field if and only if $\mathrm{rk}(\mathcal{E}) = 1$, g is Ricci flat and P as well as $\mathrm{d}Y^*$ vanish. These Einstein spaces are called plane gravitational waves (pp-waves which are Einstein spaces).

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